Solutions of Penrose's Equation

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The computational use of Killing potentials which satisfy Penrose's equation is discussed. Penrose's equation is presented as a conformal Killing-Yano equation and the class of possible solutions is analyzed. It is shown that solutions exist in spacetimes of Petrov type O, D or N. In the particular case of the Kerr background, it is shown that there can be no Killing potential for the axial Killing vector.

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I. INTRODUCTION

In a spacetime which admits a Killing vector k^a it is straightforward to find its Killing potential. Killing potentials are real bivectors Q^{ab} whose divergence returns the Killing vector, $(1/3)\nabla_b Q^{ab} = k^a$. Killing potentials attain physical importance when they are used in the Penrose-Goldberg (PG) [1] superpotential for computing conserved quantities such as mass and angular momentum. The PG superpotential is

$$U^{ab}_{PG} = \sqrt{-g}\frac{1}{2}G^{ab}{}_{cd}Q^{cd}, \tag{1}$$

where $G^{ab}{}_{cd} = -{}^*R^{*ab}{}_{cd}$, the negative right and left dual of the Riemann tensor. When the Ricci tensor is zero then $G^{ab}{}_{cd} = C^{ab}{}_{cd}$, the Weyl tensor. If Q^{ab} satisfies Penrose's equation (4) then

$$\nabla_b U^{ab}_{PG} = \sqrt{-g}G^{ab}k_b \tag{2}$$

for Einstein tensor G^{ab}. The current density

$$J^a = \sqrt{-g}G^{ab}k_b \tag{3}$$

is conserved independently of the left side of Eq.(2). It is the PG superpotential that allows the Noether quantities to be computed by integrating over closed two-surfaces, which is Penrose's quasi-local construction [2]. If one views the Killing vector itself as a conserved current then its integral over a three-surface is identically equal to 1/3 the integral of its Killing potential over the bounding two-surface and no new information can be obtained.

The tensor version of Penrose's equation [3] is

$$\mathrm{P}^{abc} := \nabla^{(a}Q^{b)c} - \nabla^{(a}Q^{c)b} + g^{a[b}Q^{c]e}{}_{;e} = 0. \tag{4}$$

With $j^a := (1/3)\nabla_b Q^{*ab}$, and $k^a := (1/3)\nabla_b Q^{ab}$, an equivalent equation [4] to $\mathrm{P}^{abc} = 0$ is

$$\nabla_c Q^{ab} = -2\delta_c^{[a} k^{b]} + 2(\delta_c^{[a} j^{b]})^*. \tag{5}$$

If Q^{ab} is a solution of the Penrose equation then $k_{(b;c)} = -(1/2)Q_{a(b}R^a{}_{c)}$ with a similar relation connecting j^a and Q^{*ab}. For Ricci-flat spacetimes j^a and k^a are Killing vectors.

For a particular spacetime the number of independent Killing vectors is between zero and ten. Penrose [3] gave the complete solution to Eq.(4) in Minkowski space for ten real independent Q^{ab}.

This work discusses the existence of Killing potentials which satisfy Penrose's equation or equivalently the conformal Killing-Yano (CKY) equation for 2-form $\mathcal{Q}$. The fact that such tensors only exist in spacetimes of Petrov type D, N or O is discussed in section III B and Appendices C and D.

In the Kerr background, it has previously been shown that there is no Killing potential for the axial Killing vector [5]. We show, in section III C, how this can be anticipated from properties of the curvature and the fact that the axial Killing vector must vanish along the axis of symmetry.

We use both the abstract index notation familiar to relativists and some coordinate free notation for which we provide Appendix A as a reference. We use bold face for index free tensor notation excepting differential forms which appear in calligraphic type. Appendix B describes some aspects of the Petrov classification in a way convenient for our purposes.

II. PREVIOUS RESULTS

An exact solution of the Penrose equation for Kerr's vacuum solution is given below in Eq.(8). This solution was first used in the context of the PG superpotential construction in [6]. The Kerr solution has two Killing vectors (KVs), stationary $k_{(t)}$ and axial $k_{(\varphi)}$, and the metric is

$$g^{Kerr} = l \otimes n + n \otimes l - m \otimes \bar{m} - \bar{m} \otimes m\,, \tag{6}$$

where $\{l, n, m, \bar{m}\}$ is the Newman-Penrose principal null coframe, given in Boyer-Lindquist coordinates by

$$\begin{aligned} l &= dt - (\Sigma/\Delta)\, dr - a \sin^2\theta\, d\varphi\,, \\ n &= \frac{\Delta}{2\Sigma}[dt + (\Sigma/\Delta)\, dr - a\sin^2\theta\, d\varphi]\,, \\ m &= \frac{1}{\sqrt{2}\bar{R}}[\mathrm{i}a\sin\theta\, dt - \Sigma\, d\theta - \mathrm{i}(r^2+a^2)\sin\theta\, d\varphi]\,, \end{aligned} \tag{7}$$

where $R = r - ia\cos\theta$, $\Sigma = R\bar{R}$ and $\Delta = r^2 + a^2 - 2m_0 r$. The Killing potential for $k_{(t)}$ is the bivector $Q^{ab}_{(t)}$ obtained by raising the components of the 2-form

$$\mathcal{Q}_{(t)} = -\left(R\mathcal{M} + \bar{R}\bar{\mathcal{M}}\right)\,, \tag{8}$$

where $\mathcal{M} := l\wedge n - m\wedge\bar{m}$ is an anti self-dual 2-form, that is $*\mathcal{M} = -i\mathcal{M}$. We mention that $Q^{ab}_{(t)}$ is a global solution since the quasi-local PG mass, resulting from integration of the PG superpotential over two-surfaces of constant t and r, is *independent of choice of two-surface*

$$\oint_{S^2} U^{ab}_{PG}\, dS_{ab} = -8\pi\ m_0 \tag{9}$$

for any r beyond the outer event horizon.

The next interesting result involves the axial Kerr symmetry. Goldberg [1] found asymptotic solutions of the Penrose equation for the Bondi-Sachs metric which includes the Kerr solution as a special case. But Glass [5] showed that the axial Killing potential could not be a solution of the Penrose equation at finite r.

The bivector $Q^{ab}_{(t)}$ generally has six independent components and so enough information to describe two Killing vectors. Since the Kerr solution has two KVs, can the dual of $Q^{ab}_{(t)}$ yield $k_{(\varphi)}$? Direct differentiation shows

$$\nabla_b Q^{*ab}_{(t)} = 0, \tag{10}$$

and so $Q^{ab}_{(t)}$ can only yield $k_{(t)}$. In fact $Q^{*ab}_{(t)}$ satisfies the Killing-Yano (KY) equation, which for an antisymmetric tensor A_{ab} can be written as

$$A_{a(b;c)} = 0. \tag{11}$$

This generalizes Killing's equation to antisymmetric tensors and can be further generalized to antisymmetric tensors of arbitrary valence. Modern usage reserves the name KY tensor for antisymmetric tensors. For the Kerr solution a symmetric tensor K_{ab} is constructed from the dual Killing potential by

$$K_{ab} = Q^{*e}_{(t)a} Q^{*(t)}_{eb} = 2\Sigma\, l_{(a} n_{b)} - r^2 g_{ab}. \tag{12}$$

This "hidden" symmetry of the Kerr solution was discovered Carter [7] and later shown to be the "square" of a two-index Killing spinor [8], or equivalently, the "square" of a Killing-Yano tensor. Though K_{ab} satisfies Eq.(11) it is symmetric and generally referred to as a Killing tensor.

Collinson [9] found that all vacuum metrics of Petrov type D, with the exception of Kinnersley's type $IIIB$, possess a KY tensor. He gave an explicit expression for both the KY tensor and it's associated Killing tensor.

III. EXISTENCE OF SOLUTIONS

A. Conformal Killing-Yano Tensors

Many of the arguments in this work depend on the conformal covariance of Penrose's equation. Penrose and Rindler [10] established the conformal covariance of its spinor form $\nabla_{A'}{}^{(A}\sigma^{BC)} = 0$ for a symmetric spinor σ^{BC}. The tensor version was previously discovered by Tachibana as the conformally covariant generalization of the KY equation [11]. In this paper it was written in the form

$$Q_{a(b;c)} = (1/3)\left[g_{bc} Q_a{}^e{}_{;e} - g_{a(b} Q_{c)}{}^e{}_{;e}\right] . \tag{13}$$

In that same work Tachibana showed that in a Ricci-flat space, for Q_{ab} a CKY bivector satisfying Eq.(13), $(1/3)\nabla^b Q_{ab}$ is a Killing vector.

From Eq.(13) we can obtain an expression for $Q_{ab;c}$ by writing out the symmetrization brackets explicitly:

$$Q_{ab;c} = -Q_{ac;b} + \frac{2}{3} g_{bc} Q_a{}^e{}_{;e} - \frac{1}{3} g_{ab} Q_c{}^e{}_{;e} - \frac{1}{3} g_{ac} Q_b{}^e{}_{;e} \, .$$

Now, since $Q_{ab;c}$ is antisymmetric in the first two indices, we have

$$\begin{aligned} 3Q_{ab;c} &= Q_{ab;c} + Q_{ab;c} - Q_{ba;c} \\ &= Q_{ab;c} - Q_{ac;b} + \frac{2}{3} g_{bc} Q_a{}^e{}_{;e} - \frac{1}{3} g_{ab} Q_c{}^e{}_{;e} - \frac{1}{3} g_{ac} Q_b{}^e{}_{;e} \\ &\qquad + Q_{bc;a} - \frac{2}{3} g_{ac} Q_b{}^e{}_{;e} + \frac{1}{3} g_{ba} Q_c{}^e{}_{;e} + \frac{1}{3} g_{bc} Q_a{}^e{}_{;e} \end{aligned}$$

and so from (13) we can deduce that

$$3Q_{ab;c} = 3Q_{[ab;c]} - 2g_{c[a} Q_{b]}{}^e{}_{;e} \, . \tag{14}$$

It is easily verified that given Eq.(14) we recover Eq.(13) and hence Eq.(14) is an alternative form of the CKY equation. Furthermore Penrose's Eq.(4) can easily be rewritten as Tachibana's Eq.(13) and so is another form of the CKY equation.

Since Q is an antisymmetric tensor, it is natural to discuss it's properties in the language of differential forms. Equation (14) is manifestly antisymmetric in the first two indices, and so it is straightforward to verify that it is the abstract index equivalent of the CKY 2-form equation of Benn *et al* [12],

$$3\nabla_Z \mathcal{Q} = Z \rfloor d\mathcal{Q} - Z^\flat \wedge \delta \mathcal{Q}, \qquad \forall Z \, . \tag{15}$$

In this form, since $*$ commutes with ∇_Z, it is readily verified using the identities given in Appendix A, that whenever $\mathcal{Q}$ is a CKY 2-form so is $*\mathcal{Q}$. Thus any solution to the CKY equation can be decomposed into self-dual and anti self-dual CKY 2-forms.

B. Existence of CKY 2-forms

On a flat background the CKY equation has many solutions, while, as will be explained, in a more general spacetime the curvature imposes tight consistency conditions and there can be at most two independent solutions, one self-dual and one anti self-dual with respect to the Hodge star. This result appears to be closely tied to the four-dimensional nature of spacetime

and the properties of these solutions are almost universally discussed in their spinor form, where the utility of the two-component spinor formalism simplifies the calculations. A detailed discussion of this can be found in spinor form in [12] or in terms of differential forms in [13].

Since any CKY 2-form can be decomposed into self-dual and anti self-dual parts that are themselves CKY 2-forms, in discussing their existence, it is sufficient to consider only 2-forms of definite Hodge-duality.

In order to understand how the curvature of the underlying spacetime restricts the solutions to Eq.(15) two steps are required. Firstly, it can be shown directly from the CKY 2-form equation that the real eigenvectors of (anti) self-dual CKY 2-forms are shear-free and hence principal null directions of the conformal tensor. Secondly, by differentiating Eq.(15) an integrability condition can be obtained that restricts the Petrov type by showing these eigenvectors to be *repeated* principal null directions.

In the case of non-null self-dual 2-forms, Dietz and Rüdiger [14] used spinor methods to obtain both of these results for a scaling covariant generalization of Eq.(15). It was later shown, again using spinor methods, that similar results can be obtained for the null case [12].

An outline of these results in the notation of differential forms is given in Appendices C and D. It is shown that apart from conformally flat spacetimes, non-null (anti) self-dual CKY 2-forms can only exist in spacetimes of Petrov type D, while null (anti) self-dual CKY 2-forms require a background spacetime of Petrov type N.

C. The divergence of a CKY 2-form

In order to apply the PG superpotential method using a given CKY 2-form $\mathcal{Q}$, its divergence (coderivative) $\delta\mathcal{Q}$ must be dual to a Killing vector. Tachibana showed that this was always the case in a Ricci flat background [11] (the result also holds for the slightly more general case of an Einstein spacetime).

In the Kerr background, there are two independent Killing vectors and two independent CKY 2-forms (one of each Hodge-duality). However the divergence of either of these CKY 2-forms is proportional to the timelike Killing vector, leaving the axial KV without a Killing potential. This allows a divergence free linear combination of the self-dual and anti self-dual CKY 2-forms to be found. The Hodge-dual of this 2-form is known as a Killing-Yano 2-form and satisfies the Killing-Yano equation (11), which can be written in a similar fashion to Eq.(15) as

$$3\nabla_X \mathcal{Q} = X \lrcorner d\mathcal{Q} \, . \tag{16}$$

However, this leaves open the question of why it is the timelike rather than the axial KV that possesses a Killing potential? To answer this question, we note that the axial Killing vector must vanish along the symmetry axis and we show that a Killing vector obtained as the divergence of a CKY 2-form must be nowhere vanishing.

First consider a non-null anti self-dual CKY 2-form $\mathcal{Q}^-$. From Eq.(15) we can write $d(\mathcal{Q}^{-2})$ in terms of $\mathcal{Q}^-$ and $\delta\mathcal{Q}^-$:

$$d\left(\mathcal{Q}^{-2}\right) = \tfrac{4}{3}\left(\delta\mathcal{Q}^-\right)^\sharp \lrcorner \mathcal{Q}^- \, ,$$

which after contracting with $\mathcal{Q}^-$ leads to

$$\delta\mathcal{Q}^- = -\tfrac{3}{2}\left(d(\mathcal{Q}^{-2})\right)^\sharp \lrcorner \mathcal{Q}^- \, .$$

Hence $\delta\mathcal{Q}^-$ vanishes if and only if $d(\mathcal{Q}^{-2})$ vanishes.

In a vacuum type D background we can deduce that $\mathcal{Q}^{-2}$ is a constant multiple of ${\Psi_2}^{-\frac{2}{3}}$ from the fact that $\mathcal{Q}^-$ is an eigen-2-form of $\mathbf{C}$ and both $\left(\mathcal{Q}^{-2}\right)^{-\frac{3}{2}}\mathcal{Q}^-$ and $\mathbf{C}\mathcal{Q}^-$ are Maxwell fields. Hence if $\mathcal{Q}^-$ vanishes, then so does Ψ_2 and the background becomes conformally flat.

Further, it can be deduced from the Bianchi identities that for a type D vacuum space-time, the gradient of Ψ_2 vanishes if and only if the Ψ_2 itself vanishes. (In the Newman-Penrose formalism, using a principal null tetrad, the vacuum type D condition implies that the only nonzero curvature component is Ψ_2 and $\kappa = \sigma = \nu = \lambda = 0$. Then, imposing

$\nabla_{X_a}\Psi_2 = 0$, the Bianchi identities lead to either $\rho = \mu = \tau = \pi = 0$ or $\Psi_2 = 0$. If we assume the former, then the NP equations for the derivatives of the spin coefficients immediately force the conclusion that Ψ_2 vanishes.) We therefore conclude that $\mathcal{Q}^{-2}$ is nowhere constant and hence $\delta\mathcal{Q}^{-}$ is nowhere vanishing and Kerr's axial Killing vector *cannot* have a Killing potential.

IV. SUMMARY

We have shown here that Penrose's equation for Killing potentials is equivalent to the conformal Killing-Yano equation for 2-forms. With no appeal to Ricci-flatness, existence of solutions was proven for spacetimes of Petrov type D, N or O. It was further shown, for type D vacuum backgrounds possessing a Killing-Yano 2-form, that Killing vectors with zeros cannot have Killing potentials.

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APPENDIX A: DIFFERENTIAL FORMS

We denote a basis for vector fields by $\{X_a\}$. The natural dual of this we denote by $\{e^a\}$, a basis for covector or 1-form fields. A coordinate basis is $X_a = \frac{\partial}{\partial x^a}$ and $e^a = dx^a$. The metric gives a natural bijection between vector and 1-form fields, which we denote by $^\sharp$ and $^\flat$; $X^\flat$ is the 1-form metric dual to the vector X and $\alpha^\sharp$ is the vector field metric dual to the 1-form α.

The 1-forms, along with the wedge product $\wedge$, generate the algebra of differential forms. The wedge product is anti-symmetric and so the differential forms of degree p can be thought

of as the subset of covariant tensors of valence p that are antisymmetric in their arguments. If α and β are 1-forms with components $\alpha_a = \alpha(X_a)$ and $\beta_a = \beta(X_a)$, then

$$\alpha \wedge \beta = \alpha_{[a}\beta_{a]} e^a \otimes e^b = \alpha_a \beta_b e^a \wedge e^b \,. \tag{A1}$$

A vector can be contracted with the p-form $\mathcal{P}$ to give a $(p-1)$-form $X \lrcorner \mathcal{P}$ so that

$$(X \lrcorner \mathcal{P})(X_{a_1}, X_{a_2}, \ldots, X_{a_{p-1}}) = p\mathcal{P}(X, X_{a_1}, X_{a_2}, \ldots, X_{a_{p-1}}) \,,$$

and so the components of a p-form can be expressed using the hook as

$$\mathcal{P}_{ab\ldots c} = \mathcal{P}(X_a, X_b, \ldots, X_c) = \frac{1}{p!} X_c \lrcorner \ldots \lrcorner X_b \lrcorner X_a \lrcorner \mathcal{P} \,.$$

We can define an inner product between any pair of 2-forms:

$$\mathcal{P} \cdot \mathcal{Q} = \frac{1}{2} X_a \lrcorner X_b \lrcorner \mathcal{P} \; X^a \lrcorner X^b \lrcorner \mathcal{Q} = 2\mathcal{P}_{ab}\mathcal{Q}^{ab} \,.$$

For $\mathcal{P} \cdot \mathcal{P}$ we write $\mathcal{P}^2$.

The metric defines a natural map from p-forms to $(n-p)$-forms called the Hodge star. In four dimensions, this maps 2-forms to 2-forms, and is defined so that

$$\mathcal{P} \wedge *\mathcal{Q} = (\mathcal{P} \cdot \mathcal{Q}) * 1 \,,$$

where $*1$ is the volume 4-form. For a Lorentzian metric, this map squares to -1 and so has eigenvalues, $\pm i$. Elements of the eigenspace corresponding to $(-i) + i$ are called (anti) self-dual 2-forms. Any 2-form can be decomposed into self-dual and anti self-dual parts

$$\mathcal{P} = \mathcal{P}^+ + \mathcal{P}^-, \qquad \text{where} \quad * \mathcal{P}^\pm = \pm i \mathcal{P} \,.$$

The Hodge star relates the hook and wedge operations by

$$X \lrcorner * \mathcal{P} = * \left(\mathcal{P} \wedge X^\flat \right) \,. \tag{A2}$$

The 2-form commutator is given by

$$[\mathcal{P}, \mathcal{Q}] = -2 X_a \lrcorner \mathcal{P} \wedge X^a \lrcorner \mathcal{Q} \tag{A3}$$

for 2-forms $\mathcal{P}$ and $\mathcal{Q}$. The Lie algebra of 2-forms under commutation is the Lie algebra of the Lorentz group.

It is often useful to work with a null coframe (basis for 1-forms) $\{l, n, m, \bar{m}\}$ dual to a Newman-Penrose tetrad, that is, one for which all inner products vanish except

$$l \cdot n = -m \cdot \bar{m} = 1\,. \tag{A4}$$

From this we can construct a basis for the anti self-dual 2-forms:

$$\mathcal{U} = -n \wedge \bar{m}, \quad \mathcal{M} = l \wedge n - m \wedge \bar{m}, \quad \mathcal{V} = l \wedge m \tag{A5}$$

with the property that all inner products vanish except

$$\mathcal{U} \cdot \mathcal{V} = 1 \qquad \text{and} \qquad \mathcal{M} \cdot \mathcal{M} = -2\,. \tag{A6}$$

In this basis, the 2-form commutator can be calculated from

$$[\mathcal{M}, \mathcal{U}] = -4\mathcal{U}\,, \quad [\mathcal{M}, \mathcal{V}] = 4\mathcal{V} \quad \text{and} \quad [\mathcal{U}, \mathcal{V}] = -\mathcal{M}\,. \tag{A7}$$

The null basis elements $\mathcal{U}$ and $\mathcal{V}$ each have one two-dimensional eigenspace, with corresponding zero eigenvalue, spanned by $\{n^\sharp, \bar{m}^\sharp\}$ and $\{l^\sharp, m^\sharp\}$ respectively. These are also the eigenspaces of $\mathcal{M}$ for which they have eigenvalues $+1$ and -1. Note that choosing $\mathcal{M}$ determines $\mathcal{U}$ and $\mathcal{V}$ up to their relative scaling or interchange.

We denote the torsion-free metric compatible covariant derivative of a 2-form $\mathcal{Q}$ with respect to a vector field Z by $\nabla_Z \mathcal{Q}$. In terms of this, the exterior derivative d and co-derivative $\delta = *d*$ can be expressed:

$$d \equiv e^a \wedge \nabla_{X_a}\,,$$

$$\delta \equiv -X^a \lrcorner \nabla_{X_a}\,.$$

APPENDIX B: THE PETROV CLASSIFICATION

In a vacuum background, the Riemann curvature tensor $\mathbf{R}$ is equal to the Weyl conformal curvature tensor $\mathbf{C}$. The symmetries of these tensors allow them to be written as the sum

of terms made of symmetric tensor products of 2-forms (i.e. terms like $\mathcal{P}\otimes\mathcal{Q}+\mathcal{P}\otimes\mathcal{Q}$). So, both can be considered as self-adjoint maps on 2-forms; if C_{abcd} are components of $\mathbf{C}$ and $\mathcal{P}_{ab}$ the components of a 2-form, then the definition

$$(\mathbf{C}\mathcal{P})_{ab} = \frac{1}{2} C_{abcd}\mathcal{P}^{cd}$$

gives the components of the 2-form $\mathbf{C}\mathcal{P}$. As a map on 2-forms, the conformal tensor preserves the eigenspaces of $*$ and so may decomposed into a part made from self-dual 2-forms alone and a part made from anti self-dual 2-forms. That is, we can write

$$\mathbf{C} = \mathbf{C}^{(+)} + \mathbf{C}^{(-)}\,,$$

where $\mathbf{C}^{(\pm)}\mathcal{Q}^{\mp} = 0$. Note that since the conformal tensor is real, $\mathbf{C}^{(-)}$ is the complex conjugate of $\mathbf{C}^{(+)}$, and so it is sufficient to classify only one of these.

The action of $\mathbf{C}^{(-)}$ on the Newman-Penrose 2-form basis described in Appendix A is the same as the action of $\mathbf{C}$ on this basis and can be written as

$$\mathbf{C}^{(-)}\begin{bmatrix}\mathcal{U}\\ \mathcal{M}\\ \mathcal{V}\end{bmatrix} = \begin{bmatrix}-\Psi_2 & \Psi_3 & -\Psi_4\\ -2\Psi_1 & 2\Psi_2 & -2\Psi_3\\ -\Psi_0 & \Psi_1 & -\Psi_2\end{bmatrix}\begin{bmatrix}\mathcal{U}\\ \mathcal{M}\\ \mathcal{V}\end{bmatrix}$$

Note that the matrix of this transformation is trace-free and the mapping is self-adjoint (that is, $\mathcal{Q}\cdot\mathbf{C}\mathcal{P} = \mathbf{C}\mathcal{Q}\cdot\mathcal{P}$).

The Petrov classification is a classification of this mapping. The spacetime is known as algebraically general when there are three distinct eigenvalues, and algebraically special otherwise. Two special cases of interest here are that of type D and N, for which a basis can be chosen so that the matrix above takes the forms,

$$\begin{bmatrix}-\Psi_2 & 0 & 0\\ 0 & 2\Psi_2 & 0\\ 0 & 0 & -\Psi_2\end{bmatrix} \quad \text{and} \quad \begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ -\Psi_0 & 0 & 0\end{bmatrix}$$

respectively.

The real null direction of a null anti self-dual 2-form $\mathcal{Q}$ is said to be a *principal null direction* (PND) of the conformal tensor if $\mathcal{Q} \cdot \mathbf{C}\mathcal{Q} = 0$. We will call such a $\mathcal{Q}$, a *principal null* (PN) 2-form. There can be at most four independent PNDs and their number and "multiplicities" provide another description of the Petrov types [3]. The multiplicities can be determined in the present formulation by the following (with $\mathcal{P}$ an anti self-dual 2-form):

multiplicity		**equivalent conditions**	
1	$\mathcal{Q} \cdot \mathbf{C}\mathcal{Q} = 0$		$\Psi_4 = 0$
2	$[\mathcal{Q}, \mathbf{C}\mathcal{Q}] = 0$	$\mathbf{C}\mathcal{Q} \propto \mathcal{Q}$	$\Psi_3 = \Psi_4 = 0$
3	$\mathcal{Q} \cdot \mathbf{C}\mathcal{P} = 0 \quad \forall \mathcal{P}$	$\mathbf{C}\mathcal{Q} = 0$	$\Psi_2 = \Psi_3 = \Psi_4 = 0$
4	$[\mathcal{Q}, \mathbf{C}\mathcal{P}] = 0 \quad \forall \mathcal{P}$	$\mathbf{C}\mathcal{P} \propto \mathcal{Q} \quad \forall \mathcal{P}$	$\Psi_1 = \Psi_2 = \Psi_3 = \Psi_4 = 0$

APPENDIX C: CKY 2-FORMS AND SHEAR-FREE CONGRUENCES

Defining the shear of a null geodesic vector field requires the choice of a "screen space", and so is not an intrinsic property of the vector field. However, if the shear vanishes for one choice of screen space, then it does for all and hence the notion of a shear-free null vector field is well defined. For definitions and discussion of optical scalars see [3].

Robinson [15] showed that the real null eigenvector l of a (anti) self-dual null 2-form ϕ is geodesic and shear-free if and only if ϕ is proportional to a source-free Maxwell field, that is $d\phi = 0$. Note that the eigenspace such a 2-form is two-dimensional, isotropic and integrable. So we can use this fact or the Frobenius integrability condition, that $d\phi = \alpha \wedge \phi$ for some α, for the vanishing of the shear of l. It is convenient here to use these results interchangeably as our criterion for a shear-free null geodesic.

Note that a shear-free null geodesic is a PND of the conformal tensor.

1. Null CKY 2-forms

Now, suppose that $\mathcal{Q}$ is a null anti self-dual CKY 2-form. Since the right hand side of CKY 2-form Eq.(15) is simply the anti self-dual part of $-2Z^\flat \wedge \delta\mathcal{Q}$, we have that

$$\begin{aligned} 0 = \mathcal{Q} \cdot 3\nabla_Z \mathcal{Q} &= -2(Z^\flat \wedge \delta\mathcal{Q}) \cdot \mathcal{Q} \\ &= 2Z \lrcorner (\delta\mathcal{Q})^\sharp \lrcorner \mathcal{Q} \,. \end{aligned}$$

Hence we can find an α such that $\delta\mathcal{Q} = \alpha^\sharp \lrcorner \mathcal{Q}$ or equivalently $d\mathcal{Q} = -\alpha \wedge \mathcal{Q}$. So the real null eigenvector of $\mathcal{Q}$ is shear-free.

2. Non-null CKY 2-forms

We wish to show that the eigenspaces of a non-null CKY 2-form $\mathcal{Q}$ are integrable and hence contain a shear-free null geodesic vector field. That is, we want to show that if X and Y are elements of the same eigenspace of $\mathcal{Q}$ with eigenvalue λ ($X \lrcorner \mathcal{Q} = \lambda X^\flat$ and $Y \lrcorner \mathcal{Q} = \lambda Y^\flat$), then so is $[X, Y]$. Since $[X, Y] = \nabla_X Y - \nabla_Y X$, we will show that $\nabla_X Y \lrcorner \mathcal{Q} = \lambda \nabla_X Y^\flat$. Note that this eigenspace is isotropic, that is $g(X, Y) = 0$.

Since the map $\alpha \mapsto \alpha^\sharp \lrcorner \mathcal{Q}$ is of maximal rank for non-null $\mathcal{Q}$, it can always be inverted and a 1-form α found such that $\delta\mathcal{Q} = -\alpha^\sharp \lrcorner \mathcal{Q}$ and $d\mathcal{Q} = \alpha \wedge \mathcal{Q}$. Using these expressions for $\delta\mathcal{Q}$ and $d\mathcal{Q}$, and the CKY 2-form Eq.(15), we have

$$\begin{aligned} \nabla_X Y \lrcorner \mathcal{Q} &= \nabla_X (Y \lrcorner \mathcal{Q}) - Y \lrcorner \nabla_X \mathcal{Q} \\ &= \lambda \nabla_X Y^\flat + X\lambda Y^\flat - \frac{1}{3}\lambda \alpha(X) Y^\flat . \end{aligned}$$

Rearranging and writing the vector equation dual to this shows that

$$(\nabla_X Y \lrcorner \mathcal{Q})^\sharp - \lambda \nabla_X Y = \Big(X\lambda - \frac{1}{3}\lambda\alpha(X) \Big) Y \,. \tag{C1}$$

Note that the right hand side is a multiple of Y and hence an eigenvector of $\mathcal{Q}$ with eigenvalue λ. However, upon contracting the left hand side with $\mathcal{Q}$, we find that it is an element of the other eigenspace, having eigenvalue $-\lambda$. Hence we must conclude that

$$\nabla_X Y \lrcorner \mathcal{Q} - \lambda \nabla_X Y^\flat = 0 \,, \tag{C2}$$

and we have the required result.

Since each eigenspace of $\mathcal{Q}$ is integrable they each give rise to a null self-dual 2-form proportional to a Maxwell field, and hence the real eigenvectors of $\mathcal{Q}$ are shear-free.

APPENDIX D: INTEGRABILITY OF CKY 2-FORMS

Apart from conformally flat spacetimes, CKY 2-forms can only exist in spacetimes of Petrov type D or N. To understand this it is sufficient to consider only CKY tensors of definite Hodge-duality, for which we give an integrability condition. For an anti self-dual CKY 2-form $\mathcal{Q}$,

$$[\mathcal{Q}, C\mathcal{P}] = \frac{1}{2}[\mathcal{P}, C\mathcal{Q}]\,, \qquad \forall \text{ 2-forms } \mathcal{P}. \tag{D1}$$

If we let $\mathcal{P} = \mathcal{Q}$, it follows that

$$[\mathbf{C}\mathcal{Q}, \mathcal{Q}] = 0.$$

Then, from the commutator algebra of anti self-dual 2-forms Eq.(A7), it can be deduced that $C\mathcal{Q}$ must be proportional to $\mathcal{Q}$, i.e.

$$\mathbf{C}\mathcal{Q} = \mu\mathcal{Q}, \tag{D2}$$

where μ is a scalar. From this, we can deduce the Petrov type as described in Appendix B.

1. Null CKY 2-forms

When $\mathcal{Q}$ is null this implies that the real null eigenvector of $\mathcal{Q}$ is a repeated principal null direction. However, if we write out Eq.(D1) in an anti self-dual 2-form basis chosen so that $\mathcal{U} = \mathcal{Q}$ and $\mathcal{V} \propto \mathcal{P}$, we find that $\mu = -\Psi_2 = 0$. Not only does this immediately tell us that $C\mathcal{Q} = 0$, but upon substitution into Eq.(D1) we have that $[\mathcal{Q}, C\mathcal{P}] = 0$ for all anti self-dual 2-forms $\mathcal{P}$. Hence the real null direction defined by $\mathcal{Q}$ is a four-fold PND and the spacetime is of Petrov type N.

2. Non-null CKY 2-forms

When $\mathcal{Q}$ is non-null, we concluded in Appendix C that the real null eigenvectors of $\mathcal{Q}$ are shear-free. If we align our anti self-dual 2-form basis so that $\mathcal{M} \propto \mathcal{Q}$ then $\mathcal{U}$ and $\mathcal{V}$ have

shear-free eigenvectors and hence are PN 2-forms. From this we conclude that $\Psi_0 = \Psi_4 = 0$. The integrability condition Eq.(D2) immediately requires that Ψ_1 and Ψ_3 vanish and hence the spacetime is of Petrov type D.

This reasoning made no use of Ricci-flatness wherein the Goldberg-Sachs theorem [16] would imply the same result.

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[1] J.N. Goldberg, Phys. Rev. **D41**, 410 (1990).

[2] R. Penrose, Proc. Roy. Soc. Lond. **A381**, 53 (1982).

[3] R. Penrose and W. Rindler, *Spinors and Space-Time*, (Cambridge Univ. Press, Cambridge, 1986), Vol 2, Eq.(6.4.6).

[4] K.P. Tod, Proc. Roy. Soc. Lond. **A388**, 457 (1983).

[5] E.N. Glass, J. Math. Phys. **37**, 421 (1996).

[6] E.N. Glass and M.G. Naber, J. Math. Phys. **35,** 4178 (1994).

[7] B. Carter, Comm. Math. Phys. **10**, 280 (1968).

[8] M. Walker and R. Penrose, Comm. Math. Phys. **18**, 265 (1970).

[9] C.D. Collinson, Int. J. Theor. Phys. **15**, 311 (1976).

[10] Ref. [3] section 6.5.

[11] S. Tachibana, Tohoku Math. J. **21**, 56 (1969).

[12] I.M. Benn, P. Charlton, and J. Kress, J. Math. Phys. **38**, 4504 (1997).

[13] J. Kress, Ph.D. Thesis, Dept. of Mathematics, Newcastle University, NSW, Australia (1998).

[14] W. Dietz and R. Rüdiger, Gen. Rel. Grav. **12**, 545 (1980).

[15] I. Robinson, J. Math. Phys. **2**, 290 (1961).

[16] J.N. Goldberg and R.K. Sachs, Acta Phys. Polon., Suppl. **22**, 13 (1962).